Steady ion momentum in nonlinear plasma waves

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The analysis of a one-dimensional two-fluid hydrodynamic model with relativistic electrons and nonrelativistic ions shows that the propagation of a nonlinear plasma wave is accompanied by a steady currentless plasma drift. Ions, due to their larger mass, appear to be the main carriers of the average momentum of the plasma wave. Two examples of nonlinear plasma waves generated by moving sources (short laser pulses and electron bunches) are analyzed to show details of the energy and momentum conservation laws.

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I. INTRODUCTION

Nonlinear plasma waves have been the object of theoretical interest in the 1950s, when some of their most remarkable properties were discovered. It was shown that the frequency of the one-dimensional nonrelativistic nonlinear plasma wave does not depend on its amplitude [1,2] (see also Ref. [3] and references therein). The dependence of the wave frequency on the amplitude results from the relativistic variation of the electron mass [4]. The amplitude of a plasma wave is limited by the wave-breaking phenomenon that occurs when the velocity of the electron fluid becomes equal to the phase velocity of the wave [4].

A new growth of interest in nonlinear plasma waves was stimulated by the idea of electron acceleration in powerful plasma waves generated by electron bunches (plasma wake-field accelerator) or by short laser pulses (laser wake-field accelerators) (see Refs. [5,6] and references therein). The main attention was attracted to the studies of the dependence of the amplitude and the period of the generated plasma wave on the parameters of the source. Usually, in these investigations, plasma is considered as an electron fluid while ions are treated as an immobile uniform background. Such an approximation seems to be reasonable but it leads to some problem concerning the average momentum of a plasma wave. Indeed, it is evident that the laser pulses as well as the electron bunches while generating plasma waves lose not only energy but also momentum. The energy lost by the source is transferred to the energy of the electron oscillations in a plasma. But the momentum lost by the source cannot be transferred into the steady momentum of the electrons, because the steady electron flow would result in the nonlinear Doppler shift of the wave frequency [7] in contradiction with the isochronism property of the nonrelativistic nonlinear plasma wave [1–3]. On the first sight it looks as if the source loses momentum but the generated wave does not gain it. Hence, to clear up the situation with the wave momentum, we consider the propagation of one-dimensional nonlinear plasma wave without the usual restriction to immobile ions. We show that independently of the generation method the propagation of the wave is accompanied by the appearance of currentless steady drift of the plasma as a whole. Due to the larger mass of ions they carry the main part of the average momentum of the plasma wave. In the particular case of the laser wake field this fact was pointed out in Ref. [8], where it was shown that the momentum lost by the laser pulse is transferred into the wake-field wave and that its average momentum is carried by ions, no matter how heavy they are.

However, in Ref. [8] this conclusion was obtained in the approximation that the velocity of the laser pulse and, hence, the phase velocity of the generated plasma wave equals the speed of light.

In this paper we take into consideration the mobility of ions and investigate the general properties of a nonlinear plasma wave propagating in a plasma with arbitrary amplitude and phase velocity less than the speed of light. To illustrate how the moving source transfers the energy and the momentum to the nonlinear plasma wave, we consider the generation of the wake field by short laser pulses and by electron bunches.

II. THE ENERGY AND MOMENTUM OF PLASMA WAVES

In our investigation we use a set of cold two-fluid hydrodynamics equations for relativistic electrons and nonrelativistic ions and restrict ourselves to the one-dimensional geometry,

$$\frac{\partial n_e}{\partial t}+\frac{\partial}{\partial z}(n_e v_e)=0, \tag{1}$$

$$\frac{\partial p_e}{\partial t}+v_e\frac{\partial p_e}{\partial z}=eE, \tag{2}$$

$$\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial z}(n_i v_i)=0, \tag{3}$$

$$\frac{\partial v_i}{\partial t}+v_i\frac{\partial v_i}{\partial z}=\frac{e_i}{m_i}E, \tag{4}$$

$$\frac{\partial E}{\partial z}=4\pi(en_e+e_i n_i), \tag{5}$$

where E is the charge separation electric field, e, m, n_e, v_e and e_i, m_i, n_i, v_i are the charges, masses, concentrations, and velocities of electrons and ions, respectively, p_e is the momentum of the electron fluid per particle, related to the electron velocity by the formula $v_e = p_e/(m\gamma)$, where

$$\gamma = \left[1 + \left(\frac{p_e}{mc}\right)^2\right]^{1/2}. \tag{6}$$

Here c is the speed of light.

From the set of Eqs. (1)–(5) one can easily obtain the energy and the momentum conservation laws,

$$\frac{\partial W}{\partial t} + \frac{\partial S}{\partial z} = 0, \tag{7}$$

$$\frac{\partial P}{\partial t} + \frac{\partial T}{\partial z} = 0, \tag{8}$$

where W, S, P, and T stand for the energy density, the energy density flow, the momentum density, and the momentum density flow, respectively,

$$W = n_e mc^2\gamma + n_i \frac{m_i v_i^2}{2} + \frac{E^2}{8\pi}, \tag{9}$$

$$S = n_e v_e mc^2\gamma + n_i v_i \frac{m_i v_i^2}{2}, \tag{10}$$

$$P = n_e p_e + m_i n_i v_i, \tag{11}$$

$$T = n_e p_e v_e + m_i n_i v_i^2 - \frac{E^2}{8\pi}. \tag{12}$$

As we are interested in one-dimensional stationary plasma waves, we suppose that all hydrodynamic variables are functions of $\xi = v_p t - z$ only, where v_p is the phase velocity of the plasma wave. In this approximation, integration of Eqs. (1)–(4) gives

$$n_e\left(\beta - \frac{v_e}{c}\right) = n_{e0}\left(\beta - \frac{v_{e0}}{c}\right), \tag{13}$$

$$\gamma - \beta\frac{p_e}{mc} = \gamma_0 - \beta\frac{p_{e0}}{mc} + \psi - \psi_0, \tag{14}$$

$$n_i\left(\beta - \frac{v_i}{c}\right) = n_{i0}\left(\beta - \frac{v_{i0}}{c}\right), \tag{15}$$

$$\left(\beta - \frac{v_i}{c}\right)^2 = -2\epsilon(\psi - \psi_0) + \left(\beta - \frac{v_{i0}}{c}\right)^2, \tag{16}$$

where we assumed that the density and the velocity of electrons and ions equal n_{e0}, v_{e0}, n_{i0}, and v_{i0}, respectively, at the point where $\psi = \psi_0$. At the same point γ and p_e attain the values denoted as $\gamma_0 = [1 + (p_{e0}/mc)^2]^{1/2}$, and $p_{e0} = mv_{e0}/(1 - v_{e0}^2/c^2)^{-1/2}$, respectively. Here $\beta = v_p/c$ is the dimensionless phase velocity, $\epsilon = (Zm/m_i) \ll 1$ is a small parameter, Z is the ion charge number, and ψ is the electrostatic potential determined by the relation

$$\frac{eE}{mc^2} = -\frac{d\psi}{d\xi}. \tag{17}$$

To establish the relations between n_{e0}, n_{i0}, v_{e0}, and v_{i0}, let us assume that in Eqs. (13)–(16) the potential is a constant, $\psi = \psi_0$, and that the perturbations in the plasma are absent. We find that in the unperturbed plasma $v_e = v_{e0}$, $n_e = n_{e0}$, $v_i = v_{i0}$, and $n_i = n_{i0}$. Assuming that in unperturbed state the plasma is neutral and electrons and ions are at rest we obtain that $n_{e0} = Zn_{i0}$ and $v_{e0} = v_{i0} = 0$. By means of these relations Eqs. (13)–(16) are much simplified. One can easily see that the electrons and the ions stop at the same points where the potential of the plasma wave is equal to ψ_0. Without the loss of generality, we may put $\psi_0 = 1$.

The set of Eqs. (13)–(16) allows us to express all the hydrodynamic variables as functions of the potential ψ

$$\gamma = \gamma_p^2[\psi - \beta(\psi^2 - \gamma_p^{-2})^{1/2}], \tag{18}$$

$$\frac{n_e}{n_{e0}} = \frac{\beta\gamma_p^2[\psi - \beta(\psi^2 - \gamma_p^{-2})^{1/2}]}{(\psi^2 - \gamma_p^{-2})^{1/2}}, \tag{19}$$

$$\frac{p_e}{mc} = \gamma_p^2[\beta\psi - (\psi^2 - \gamma_p^{-2})^{1/2}], \tag{20}$$

$$\frac{n_i}{n_{i0}} = \frac{\beta}{\sqrt{\beta^2 - 2\epsilon(\psi - 1)}}, \tag{21}$$

$$\frac{v_i}{c} = \beta - \sqrt{\beta^2 - 2\epsilon(\psi - 1)}, \tag{22}$$

where $\gamma_p = (1 - \beta^2)^{-1/2}$ is the relativistic factor determined by the phase velocity of the plasma wave.

Substituting n_e and n_i from Eqs. (19) and (21) into the Poisson equation (5) we obtain

$$\frac{d^2\psi}{d\eta^2} = \frac{\beta\gamma_p^2[\psi - \beta(\psi^2 - \gamma_p^{-2})^{1/2}]}{(\psi^2 - \gamma_p^{-2})^{1/2}} - \frac{\beta}{\sqrt{\beta^2 - 2\epsilon(\psi - 1)}}, \tag{23}$$

where $\eta = k_p\xi$, $k_p = \omega_p/c$, and $\omega_p = (4\pi e^2 n_{e0}/m)^{1/2}$ is the plasma frequency. The nonrelativistic limit of Eq. (23) was used in Ref. [9] to study the influence of the ion motion on the plasma wave frequency. For the case of immobile ions $(\epsilon = 0)$, Eq. (23) coincides with the well-known and much studied one (see, for example, Refs. [10–15]). The Poisson equation, taking into account relativistic motion of electrons and ions, was derived in Refs. [16,17].

The first integral of this equation may be written in the form

$$\frac{1}{2}\left(\frac{d\psi}{d\eta}\right)^2 = I_0 - \beta\gamma_p^2[\beta\psi - (\psi^2 - \gamma_p^{-2})^{1/2}] - \frac{\beta}{\epsilon}[\beta - \sqrt{\beta^2 - 2\epsilon(\psi - 1)}], \tag{24}$$

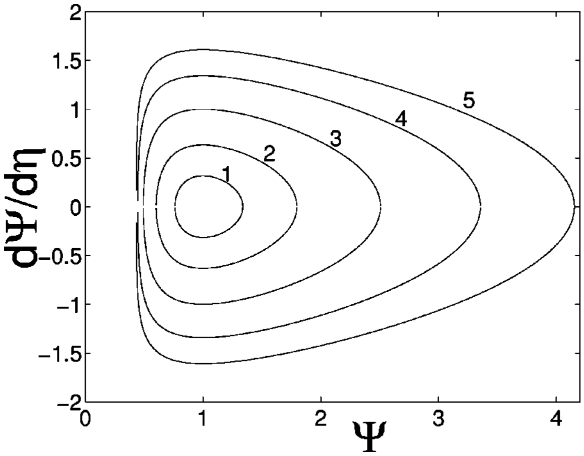

FIG. 1. Phase portraits of Eq. (25) showing $d\psi/d\eta$ as a function of ψ for $\beta=0.9$ and several values of integration constant I_0. $1-I_0=0.05$; $2-I_0=0.2$, $3-I_0=0.5$; $4-I_0=0.9$; $5-I_0=1.294$.

where I_0 is the integration constant. Expanding Eqs. (23) and (24) in power series of small parameter ϵ to the zeroth-order terms, we obtain

$$\frac{d^2\psi}{d\eta^2}=\gamma_p^2\frac{[\beta\psi-(\psi^2-\gamma_p^{-2})^{1/2}]}{(\psi^2-\gamma_p^{-2})^{1/2}}. \tag{25}$$

$$\frac{1}{2}\left(\frac{d\psi}{d\eta}\right)^2=I_0-U_0(\psi), \tag{26}$$

where

$$U_0(\psi)=\gamma_p^2[\psi-\beta(\psi^2-\gamma_p^{-2})^{1/2}]-1. \tag{27}$$

The function $U_0(\psi)$ is real only for $\psi\geqslant\gamma_p^{-1}$. In this region it has its only extremum (the minimum) located at $\psi=1$, where $U_0(1)=0$. According to Eq. (26), the amplitude of the plasma wave electric field at $\psi=1$ attains a maximum value equal to

$$E_{\max}=\frac{mc\omega_p}{|e|}\sqrt{2I_0}. \tag{28}$$

However, the value of I_0 cannot be arbitrarily large. The periodic solutions to Eq. (26), describing the nonlinear plasma waves, exist only if the right-hand side of Eq. (26) takes real positive values. Therefore, the magnitude of I_0 is limited by the condition $I_0\leqslant U_0(\gamma_p^{-1})$ from which we find

$$I_{0,\max}=\gamma_p-1. \tag{29}$$

This magnitude being substituted into Eq. (28) gives the well-known result of Ref. [4] for the wave-breaking amplitude of plasma waves E_{wb}. The influence of the ion motion on the wave-breaking field is proportional to the small factor ϵ [17].

Figure 1 shows a typical phase portrait of Eq. (25), with $d\psi/d\eta$ plotted as a function of ψ for several values of integration constant I_0. The dimensionless phase velocity β is taken to be 0.9. The curve with the largest value of I_0 corresponds to its maximal permissible value for $\beta=0.9$ as determined by Eq. (29). It bounds the region of possible real solutions of Eq. (25).

Using Eqs. (18)–(22), we can express all hydrodynamic variables appearing in the conservation laws in terms of the electrostatic potential ψ. For example, the dimensionless energy density can be written as

$$U=\frac{W}{n_{e0}mc^2}=\frac{\beta\gamma^2}{(\psi^2-\gamma_p^{-2})^{1/2}}+\frac{1}{2}\left(\frac{d\psi}{d\eta}\right)^2-1+\beta\frac{[\beta-\sqrt{\beta^2-2\epsilon(\psi-1)}]^2}{2\epsilon\sqrt{\beta-2\epsilon(\psi-1)}}. \tag{30}$$

Note that, while writing this expression, we subtracted the rest energy of electrons $n_{e0}mc^2$ from the total energy. Analogously, we can write expressions for dimensionless energy density flow N, momentum density Q, and momentum density flow G

$$N=\frac{S}{n_{e0}mc^3}=\frac{\gamma(\gamma-\psi)}{(\psi^2-\gamma_p^{-2})^{1/2}}+\beta\frac{[\beta-\sqrt{\beta^2-2\epsilon(\psi-1)}]^3}{2\epsilon\sqrt{\beta^2-2\epsilon(\psi-1)}}. \tag{31}$$

$$Q=\frac{P_e+P_i}{n_{e0}mc}=Q_e+Q_i=\frac{\gamma(\gamma-\psi)}{(\psi^2-\gamma_p^{-2})^{1/2}}+\beta\frac{\beta-\sqrt{\beta^2-2\epsilon(\psi-1)}}{\epsilon\sqrt{\beta^2-2\epsilon(\psi-1)}}. \tag{32}$$

$$G=\frac{T}{mc^2n_{e0}}=\frac{(\gamma-\psi)^2}{\beta(\psi^2-\gamma_p^{-2})^{1/2}}-\frac{1}{2}\left(\frac{d\psi}{d\eta}\right)^2+\frac{[\beta-\sqrt{\beta^2-2\epsilon(\psi-1)}]^2}{\epsilon\sqrt{\beta^2-2\epsilon(\psi-1)}}. \tag{33}$$

The last terms of Eqs. (30)–(33) describe the ion contribution to the corresponding quantities. Expanding these terms in power series of the small parameter ϵ, we find that these contributions are small everywhere except for Eq. (32). In the limit $\epsilon\ll1$ the ion momentum Q_i remains finite and equals $\beta^{-1}(\psi-1)$. Thus, we conclude that ions effect is essential only for the momentum density of the plasma wave, giving a finite contribution even in the limit of infinitely heavy ions. For all the other hydrodynamic quantities, as well as in the Poisson equation, the influence of the ions may be entirely neglected. As a result, Eqs. (30)–(33) take the form

$$U=\beta\gamma_p^4\frac{[\psi-\beta(\psi^2-\gamma_p^{-2})^{1/2}]^2}{(\psi^2-\gamma_p^{-2})^{1/2}}-\gamma_p^2[\psi-\beta(\psi^2-\gamma_p^{-2})^{1/2}]+I_0, \tag{34}$$

$$N=\beta\gamma_p^4\frac{[\psi-\beta(\psi^2-\gamma_p^{-2})^{1/2}][\beta\psi-(\psi^2-\gamma_p^{-2})^{1/2}]}{[(\psi^2-\gamma_p^{-2})^{1/2}]}, \tag{35}$$

$$Q=Q_e+Q_i$$
$$=\beta\gamma_p^4\frac{[\psi-\beta(\psi^2-\gamma_p^{-2})^{1/2}][\beta\psi-(\psi^2-\gamma_p^{-2})^{1/2}]}{(\psi^2-\gamma_p^{-2})^{1/2}}$$
$$+\beta^{-1}(\psi-1), \tag{36}$$

$$G=\beta\gamma_p^4\frac{[\beta\psi-(\psi^2-\gamma_p^{-2})^{1/2}]^2}{(\psi^2-\gamma_p^{-2})^{1/2}}+\gamma_p^2[\psi-\beta(\psi^2-\gamma_p^{-2})^{1/2}]$$
$$-1-I_0. \tag{37}$$

We emphasize once again that in these equations the only term appearing due to the ions is $\beta^{-1}(\psi-1)$ in Eq. (36).

All quantities characterizing the nonlinear plasma wave may be presented as the sum of two terms. One of them is independent of η and determines the average of the corresponding quantity. The other one is purely oscillatory and the integral of it over the plasma wavelength is zero.

To find the average parts of the quantities (34)–(37) we calculate initially the wavelength of the plasma wave λ, which equals twice the distance between the two nearest minimum and maximum points of the potential ψ. In k_p^{-1} units it has the following form:

$$\lambda=2\int_{\psi_-}^{\psi_+}\frac{d\psi}{d\psi/d\eta}, \tag{38}$$

where $d\psi/d\eta$ is determined by Eq. (26), while ψ_+ and ψ_- are the roots of the right-hand side of the equation $d\psi/d\eta=0$,

$$\psi_\pm=1+I_0\pm\beta\sqrt{(1+I_0)^2-1}. \tag{39}$$

Introducing a new integration variable $y=\gamma$, we can rewrite Eq. (38) in the form

$$\lambda=2\sqrt{2}\beta\int_1^{1+I_0}\frac{y\,dy}{\sqrt{(1+I_0-y)(y^2-1)}}. \tag{40}$$

Note, this integral as well as other integrals below may be reduced to elliptic integrals [18].

It can be shown that the average energy density flow [Eq. (35)], the electron part of the average momentum density in Eq. (36), and the average current density are equal to zero. To prove this statement, let us consider at first the energy density flow (35). The expression for N can be written in the form

$$N=\frac{d}{d\eta}\left[(1+I_0)\frac{d\psi}{d\eta}-\frac{1}{6}\left(\frac{d\psi}{d\eta}\right)^3\right]. \tag{41}$$

The integration of Eq. (41) over the plasma wavelength involves the quantity $d\psi/d\eta$ taken at the extremum points $\psi_\pm$ where it equals zero. We obtain the evident conclusion that in the cold plasma model the average energy density flow of the nonlinear plasma wave is equal to zero. According to Eqs. (36) and (35) the electron part of the momentum density Q_e is equal to N and hence, the average electron momentum density also equals zero. Using continuity equations (1) and (3), the current density may be rewritten in the form

$$J=\frac{en_cv_e+e_in_iv_i}{\epsilon n_{e0}c}=\beta\frac{d^2\psi}{d\eta^2}. \tag{42}$$

It is evident that the average current density is also equal to zero. This relation is valid independently of the parameter ϵ. It is worth noting that although the average electron current density and the average ion current density may differ from zero, their sum always equals zero. So, the current density, the electron momentum density, and the energy density flow of plasma waves are purely oscillatory functions.

From Eq. (34) we find for the average energy density of plasma waves

$$\langle U\rangle=\frac{2}{\lambda}\int_{\psi_-}^{\psi_+}\frac{U(\psi)d\psi}{d\psi/d\eta}=I_0. \tag{43}$$

Averaging Eq. (36) by means of the same procedure as above we obtain

$$\langle G\rangle=\frac{2\sqrt{2}\beta}{\lambda}\int_1^{1+I_0}\frac{(2y^2-(1+I_0)y-1)dy}{\sqrt{(1+I_0-y)(y^2-1)}}. \tag{44}$$

At last, the average momentum density is equal to

$$\langle Q\rangle=\langle Q_i\rangle=\langle\beta^{-1}(\psi-1)\rangle$$
$$=\frac{2\sqrt{2}}{\lambda}\int_1^{1+I_0}-\frac{(2y^2-y-1)y\,dy}{\sqrt{(1+I_0-y)(y^2-1)}}. \tag{45}$$

Thus, the average energy density of the plasma wave is equal to the value I_0, which appears as an integration constant in Eq. (24). We see also that the average energy density and the average momentum density flow are connected with electrons, while the average momentum density appears mainly due to ions.

Keeping in mind that the temporal and the spatial dependance of all quantities only relies on $\eta=k_p(v_pt-z)$, we find from the conservation laws (7) and (8),

$$\beta U-N=C_1, \tag{46}$$

$$\beta Q-G=C_2, \tag{47}$$

where C_1 and C_2 are integration constants. Averaging these equations and taking into account that $\langle N\rangle=0$, we obtain

$$\beta\langle U\rangle=C_1, \tag{48}$$

$$\beta\langle Q\rangle-\langle G\rangle=C_2. \tag{49}$$

Substituting $\langle U\rangle$, $\langle Q\rangle$, and $\langle G\rangle$ from Eqs. (43)–(45) we find the relations between the integration constants,

$$C_1=\beta I_0, \tag{50}$$

$$C_2=I_0. \tag{51}$$

It should be emphasized that $\langle Q\rangle$ and $\langle G\rangle$ in general case are rather complicated functions of I_0, but they always satisfy Eq. (49). In the limit $I_0 \ll 1$, Eqs. (44) and (45) take the forms

$$\beta\langle Q\rangle = \frac{3}{2} I_0, \tag{52}$$

$$\langle G\rangle = \frac{1}{2} I_0. \tag{53}$$

It is evident that these equations are in full agreement with Eq. (49).

In this analysis, we assumed that the nonlinear plasma wave had some given amplitude determined by the constant I_0 and were not interested in the process of the wave excitation. Now, on the basis of the general relations obtained above we consider the conservation laws for the plasma waves that are generated by two different localized sources propagating into plasmas.

III. GENERATION OF PLASMA WAVES BY SHORT LASER PULSES

The low-frequency (as compared to the laser frequency) plasma response to the laser pulse action may be described by means of the set of hydrodynamic equations (see, for example, Ref. [19]) consisting of Eqs. (1), (3)–(5), and the modified equation of the electron fluid motion

$$\frac{\partial p_e}{\partial t} + v_e \frac{\partial p_e}{\partial z} = eE - \frac{mc^2}{4\gamma} \frac{\partial |a|^2}{\partial z}, \tag{54}$$

where $v_e = p_e/(m\gamma)$, and γ is defined as

$$\gamma = \left[1 + \left(\frac{p_e}{mc}\right)^2 + \frac{|a|^2}{2}\right]^{1/2}. \tag{55}$$

Here a is the slowly varying amplitude (the envelope) of the dimensionless high-frequency electron momentum in the laser field. The high-frequency ion motion is entirely neglected.

From the set of Eqs. (1), (3)–(5), and (54), the energy and momentum conservation laws follow [8]

$$\frac{\partial W}{\partial t} + \frac{\partial S}{\partial z} = \frac{mc^2 n_e}{4\gamma} \frac{\partial |a|^2}{\partial t}, \tag{56}$$

$$\frac{\partial P}{\partial t} + \frac{\partial T}{\partial z} = -\frac{mc^2 n_e}{4\gamma} \frac{\partial |a|^2}{\partial z}, \tag{57}$$

where W, S, P, and T are defined by Eqs. (9)–(12). The right-hand side terms in the conservation laws determine the energy and the momentum that the laser pulse loses to generate the low-frequency plasma response [8].

If the laser pulse is short enough that its energy and shape change insignificantly over the time of its duration, we can consider all the variables, characterizing the plasma response, as functions of the variable $\xi = v_g t - z$ only, where v_g is the group velocity of the laser pulse (the so-called quasistatic approximation [20]). In contrast with the analysis of Ref. [8], we do not neglect the difference between the group velocity of the pulse and the speed of light c. Thus we can express all hydrodynamic variables as functions of the potential ψ,

$$\gamma = \gamma_p^2 [\psi - \beta\{\psi^2 - \gamma_p^{-2}(1+|a|^2/2)\}^{1/2}], \tag{58}$$

$$\frac{n_e}{n_{e0}} = \frac{\beta\gamma_p^2[\psi - \beta\{\psi^2 - \gamma_p^{-2}(1+|a|^2/2)\}^{1/2}]}{\{\psi^2 - \gamma_p^{-2}(1+|a|^2/2)\}^{1/2}}, \tag{59}$$

$$\frac{p_e}{mc} = \gamma_p^2[\beta\psi - \{\psi^2 - \gamma_p^{-2}(1+|a|^2/2)\}^{1/2}], \tag{60}$$

where $\gamma_p = 1/\sqrt{1-\beta^2}$ and $\beta = v_g/c$. In the case we are considering now, the phase velocity of the plasma wave v_p is equal to the group velocity of the laser pulse v_g. The expressions (21) and (22) for n_i and v_i remain valid in the problem under consideration.

Substituting n_e and n_i into the Poisson equation we obtain

$$\frac{d^2\psi}{d\eta^2} = \frac{\beta\gamma_p^2[\psi - \beta\{\psi^2 - \gamma_p^{-2}(1+|a|^2/2)\}^{1/2}]}{\{\psi^2 - \gamma_p^{-2}(1+|a|^2/2)\}^{1/2}} - \frac{\beta}{\sqrt{\beta^2 - 2\epsilon(\psi - 1)}}. \tag{61}$$

The ultrarelativistic limit $(\beta = 1)$ of this equation was studied in Ref. [8]. For the case of immobile ions $(\epsilon = 0)$, Eq. (61) was investigated in Refs. [21,22]. (See also Ref. [23] where the Poisson equation is considered including both relativistic ion effects and high-frequency ion motion.)

Keeping in mind that the laser pulse is absent at $\eta \to -\infty$, we integrate Eq. (61) for a given $|a|^2$ and obtain

$$\frac{1}{2}\left(\frac{d\psi}{d\eta}\right)^2 = -\beta\gamma_p^2[\beta\psi - \sqrt{\psi^2 - \gamma_p^{-2}(1+|a|^2/2)}] - \frac{\beta}{\epsilon}[\beta - \sqrt{\beta^2 - 2\epsilon(\psi-1)}] + \frac{\beta}{4}\int_{-\infty}^{\eta} \frac{d\eta' \, d|a|^2/d\eta'}{\sqrt{\psi^2 - \gamma_p^{-2}(1+|a|^2/2)}}. \tag{62}$$

An analytical solution may be found for the square shaped laser pulse with $|a|^2 = \text{const}$ between the leading and trailing edges of pulse. For such a pulse form in the immobile ion approximation, the generation of plasma waves and wave-breaking phenomenon were investigated in Refs. [21,22]. In particular, in Ref. [22] it was shown that the wave-breaking electric field amplitude of the plasma wave in the pulse region is higher than that behind the pulse in the wake-field region.

In the limit $\epsilon \ll 1$ behind the laser pulse $(\eta \to \infty)$, Eq. (62) takes a form identical to Eq. (26), where instead of the parameter I_0, determining the amplitude of the plasma wave, stands the quantity

$$I_p = \frac{\beta}{4}\int_{-\infty}^{\infty}\frac{d\eta' d|a|^2/d\eta'}{\sqrt{\psi^2-\gamma_p^{-2}(1+|a|^2/2)}}, \tag{63}$$

that for once is determined by the intensity and the form of the laser pulse as well as by the potential variation inside the pulse. In accordance with the determination (43) the quantity (63) characterizes the average energy density of the plasma wake wave.

As we have reduced the problem to Eq. (26), all the results obtained in the preceding section concerning the plasma wave properties remain valid for the laser wake field.

Particularly, integrating conservation laws (56) and (57) in the quasistatic approximation and averaging results, we obtain in the wake-field region the equations analogous to Eqs. (48) and (49) where C_1 and C_2 are equal to βI_p and I_p, respectively. These quantities describe the energy and the momentum transfer from the laser pulse to the plasma. Note that in a strong nonlinear plasma wave each term of the left-hand side of Eq. (49) is a complicated function of I_p, but the difference of their values always equals I_p.

As in the preceding section, I_p cannot exceed the threshold value given by the right-hand side of Eq. (29), above which the wave breaking takes place. In the square shaped pulse region this phenomenon was investigated in Ref. [22].

IV. GENERATION OF PLASMA WAVES BY ELECTRON BUNCHES

In this section we consider plasma waves generated by an electron bunch whose density is much smaller than that in the background plasma. To investigate the wake-field generation by one-dimensional electron bunches, it is necessary to take into account the bunch charge in the Poisson equation (see, for example, Ref. [6] and references therein)

$$\frac{\partial E}{\partial z} = 4\pi(en_b + en_e + e_i n_i), \tag{64}$$

where n_b is the electron bunch density, which is considered as given and unchangeable. As for the other hydrodynamic quantities, they satisfy Eqs. (1)–(4). These equations together with the Poisson equation (64) lead now to the conservation laws in the form

$$\frac{\partial W}{\partial t} + \frac{\partial S}{\partial z} = -ev_b n_b E, \tag{65}$$

$$\frac{\partial P}{\partial t} + \frac{\partial T}{\partial z} = -en_b E, \tag{66}$$

where v_b is the velocity of the electron bunch and W, S, P, and T are defined by Eqs. (9)–(12). The right-hand side terms of these equations determine, respectively, the amount of the work executed by the bunch over the plasma and the force acting on the plasma. In the quasistatic approximation and upon condition $\epsilon \ll 1$, the Poisson equation takes the form

$$\frac{d^2\psi}{d\eta^2} = N_b + \gamma_p^2\frac{[\beta\psi - (\psi^2-\gamma_p^{-2})^{1/2}]}{(\psi^2-\gamma_p^{-2})^{1/2}}, \tag{67}$$

where $N_b = n_b/n_{e0} \ll 1$ and $\beta = v_b/c$ are the dimensionless density and the velocity of the bunch, $\gamma_p = \sqrt{1-\beta^2}$. The first integral of this equation is

$$\frac{1}{2}\left(\frac{d\psi}{d\eta}\right)^2 = 1 + \int_{-\infty}^{\eta} d\eta' N_b(\eta')\frac{d\psi}{d\eta'} - \gamma_p^2[\psi - \beta(\psi^2-\gamma_p^{-2})^{1/2}]. \tag{68}$$

Behind the bunch ($\eta \to \infty$), Eq. (68) takes a form identical to Eq. (26) where I_0 is replaced by I_b,

$$I_b = \int_{-\infty}^{\infty} d\eta' N_b(\eta')\frac{d\psi}{d\eta'}. \tag{69}$$

The magnitude of I_b depends on the electron bunch form and on the variation of the potential ψ inside the bunch.

So, the problem is reduced again to Eq. (26). Hence, all the results obtained in the previous sections concerning the plasma wave properties remain valid. The value of I_b determines all characteristics of the wave generated in the wake region. The maximum value of I_b is determined by the wave-breaking threshold and is given by the right-hand side of Eq. (29).

V. CONCLUSIONS

The nonlinear plasma wave excitation transfers into the plasma both an average momentum and an average energy. The average energy is contained mainly in the plasma electrons oscillating in the self-consistent electric field. The average momentum, in the cold plasma approximation, appears in the form of a steady currentless plasma drift as a whole. Owing to the larger mass, ions are the main carriers of the average momentum of a nonlinear plasma wave. In some sense, this effect is similar to so-called acoustic wind, the steady mass transport in a nonlinear sound wave (see, for example, Ref. [24]).

This result may be extended to the case of a hot plasma. Multiplying the electron and the ion continuity equations by the electron and ion charges, respectively, and summing the obtained equations lead to the charge continuity equation. Integrating it over the plasma wavelength and keeping in mind that in the initially neutral plasma the total charge over the plasma wavelength is equal to zero, one can see that in an initially currentless plasma the average current density also equals zero independently of the plasma temperature. This means that the absolute values of the average electron and ion current densities are equal to each other. Since the average electron (ion) momentum density differs from the average electron (ion) current density by a multiplier proportional to the electron (ion) mass, it is evident that the average ion momentum density is much larger than the average electron momentum density except in the ultrarelativistic case, when the effective electron and ion masses may be comparable. So, we may conclude that even in a hot plasmas the average momentum density of plasma waves is carried mostly by ions.

Note, the plasma drift occurs with respect to the coordinate frame connected with the rest plasma in the absence of the wave. In the case of a plasma wave excited by means of localized sources (laser pulses or electron bunches) the reference frame is connected with the immobile plasma in front of the source. The drift velocity depends on the ratio m/m_i and the plasma wave amplitude. In general case the latter dependence is rather complicated [see Eq. (45)] but it becomes quite simple in the limit of small plasma wave amplitude when $I_0 \ll 1$ ($|e|E_{\max} \ll mc\omega_p$). In such a limit we find from Eqs. (28), (32), and (52) the plasma drift velocity $u = \langle v_i \rangle$ as

$$\frac{u}{c} = \frac{3}{4}\frac{Zm}{m_i}\left(\frac{|e|E_{\max}}{mc\omega_p^2}\right)^2. \tag{70}$$

The appearance of the plasma drift as well as the other nonlinear effects due to the ion mobility are responsible for an additional nonlinear frequency shift of a plasma wave. This problem was investigated in Ref. [9] for the case of plasma waves with nonrelativistic phase velocities ($\beta \ll 1$). In the Appendix we calculate the nonlinear plasma wave frequency shift including the ion effects in the linear approximation for an arbitrary wave phase velocity. It has the form

$$\frac{\omega_l - \omega_p}{\omega_p} = -\frac{3}{8}I_0 + \frac{\epsilon}{2} + \frac{15}{16}\epsilon I_0\left(1 + \frac{2}{\beta^2}\right), \tag{71}$$

where ω_l is the modified plasma frequency. Here the first term arises from the relativistic electron mass variation [4]; the second term is a consequence of the reduced mass $mm_i/(m+m_i)$ characterizing the frequency of the linear oscillations of plasma with mobile ions; the third term results from the nonlinearity of the plasma wave that is connected with the ion mobility. To conform our result to Ref. [9] it is necessary to replace I_0 by $d_0^2\beta^2/4$, where d_0 is the amplitude of the plasma wave in Ref. [9], and consider the limit $\beta = 0$. For a relativistic plasma wave with a phase velocity of the order of the speed of light ($\beta \simeq 1$) the dimensionless nonlinear ion correction to the plasma frequency is $(45/16)\epsilon I_0$.

To evaluate the plasma drift velocity, we consider the plasma wave with the amplitude $E_{\max} = 2\times 10^8$ V/cm propagating in a hydrogen plasma with a density 10^{17} cm^{-3} ($I_0 = 1/2$). From Eq. (70) we find that $u \simeq 1.2\times 10^7$ cm/s. In principle, this velocity may be detected by means of scattering of a probe laser beam.

APPENDIX

In what follows, we investigate the influence of the ion mass finiteness on the frequency of nonlinear plasma oscillations. In the linear approximation with respect to the small parameter ϵ determined by the electron to ion mass ratio we find from Eq. (24)

$$\frac{1}{2}\left(\frac{d\psi}{d\eta}\right)^2 = I_0 + 1 - \gamma_p^2[\psi - \beta(\psi^2 - \gamma_p^{-2})^{1/2}] - \frac{\epsilon}{2\beta^2}(\psi - 1)^2. \tag{A1}$$

Then the dimensionless plasma wavelength (38) is given by

$$\lambda' = \sqrt{2}\int_{\psi'_-}^{\psi'_+} d\psi \left\{ I_0 + 1 - \gamma_p^2[\psi - \beta(\psi^2 - \gamma_p^{-2})^{1/2}] - \frac{\epsilon}{2\beta^2}(\psi - 1)^2 \right\}^{-1/2}, \tag{A2}$$

where the prime in the left-hand side of the equation indicates that Eq. (A2) contains the ion term unlike Eq. (40), where this term is omitted. The minimum ψ'_- and maximum ψ'_+ of the potential ψ are determined from the equation

$$I_0 + 1 = \gamma_p^2[\psi'_\pm - \beta(\psi'^2_\pm - \gamma_p^{-2})^{1/2}] + \frac{\epsilon}{2\beta^2}(\psi'_\pm - 1)^2. \tag{A3}$$

Introducing in Eq. (A2) a new variable of integration $x = \gamma_p^2[\psi - \beta(\psi^2 - \gamma_p^{-2})^{1/2}]$, we rewrite the formula for the plasma wavelength in the form $\lambda' = \lambda_+ - \lambda_-$, where

$$\lambda_\pm = \sqrt{2}\int_1^{x_\pm} dx \left[1 \pm \frac{\beta x}{\sqrt{x^2-1}}\right]\left[(x_\pm - x) + \frac{\epsilon}{2\beta^2}\{(x_\pm - 1 \pm \beta\sqrt{x_\pm^2 - 1})^2 - (x - 1 \pm \beta\sqrt{x^2-1})^2\}\right]^{-1/2}. \tag{A4}$$

Taking into consideration the condition $\epsilon \ll 1$, we find from Eq. (A3) for the upper limits of integration in Eq. (A4)

$$x_\pm = 1 + I_0 - \frac{\epsilon}{2\beta^2}(I_0 \pm \beta\sqrt{2I_0 + I_0^2})^2. \tag{A5}$$

Using the same condition $\epsilon \ll 1$, we present Eq. (A4) in the form

$$\lambda_\pm = \sqrt{2}\int_1^{x_\pm} dx \left(1 \pm \frac{\beta x}{\sqrt{x^2-1}}\right)\frac{1}{\sqrt{x_\pm - x}} \times\left[1 - \frac{\epsilon}{2\beta^2(x_\pm - x)}\{(x_\pm - 1 \pm \beta\sqrt{x_\pm^2-1})^2 - (x - 1 \pm \beta\sqrt{x^2-1})^2\}\right]. \tag{A6}$$

In general case the integrals in Eq. (A6) are quite complicated functions of I_0. To simplify our calculations, suppose that $I_0 < 1$. Then from Eq. (A5) we obtain

$$x_\pm = 1 + I_0(1+\epsilon) \pm \frac{\sqrt{2}\epsilon}{\beta} I_0^{3/2}. \tag{A7}$$

Keeping in mind that in the case of small I_0 the integration region in Eq. (A6) is localized near the unity, we find from Eq. (A6) after some routine calculations

$$\lambda' = 2\pi\beta\left[1+\frac{3}{8}I_0-\frac{\epsilon}{2}-\frac{15}{16}I_0\epsilon\left(1+\frac{2}{\beta^2}\right)\right]. \qquad (A8)$$

The dimensionless plasma wavelength λ' is connected with the modified plasma frequency ω_l by means of the formula $\lambda' = 2\pi\beta\omega_p/\omega_l$ that leads to Eq. (71).

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